Dispersion

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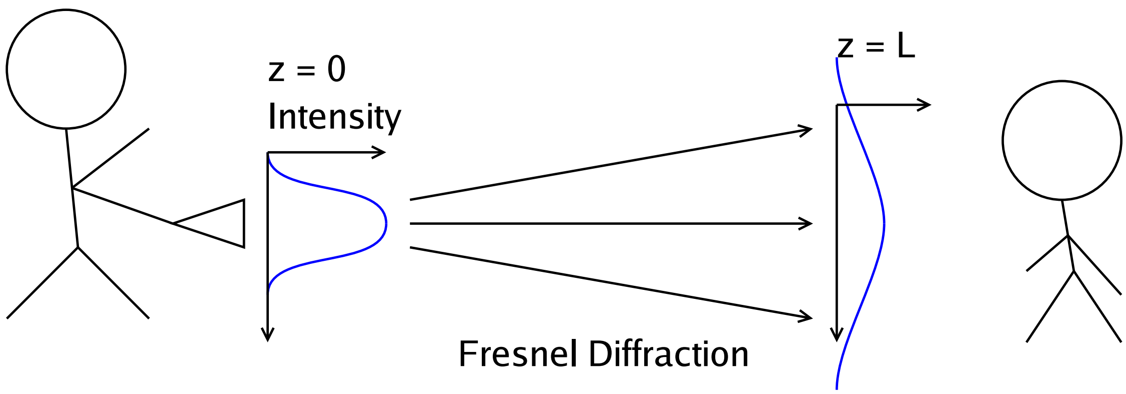
DIFFRACTION



Before learning about dispersion, it is nice to review the concepts of diffraction. Imagine that I have a nice Gaussian beam coming out of a laser, say at $z = 0$, the electric field is given by

$$E(x, y, z = 0, t) = E_0 \exp(-\frac{x^2 + y^2}{2W_0^2}) \exp(-j\omega_0 t) \tag{1}$$

From the the wave equation, We know that the *eigenmodes* of the free space are plane waves,

$$E_i(x, y, z, t) = \tilde{E}(k_x, k_y) \exp(jk_x x + jk_y y + j\sqrt{k_0^2 - k_x^2 - k_y^2} z) \exp(-j\omega_0 t) \tag{2}$$

And by the paraxial approximation

$$E_i(x, y, z, t) \approx \tilde{E}(k_x, k_y) \exp(jk_x x + jk_y y + jk_0 z - j\frac{k_x^2 + k_y^2}{2k_0} z) \exp(-j\omega_0 t) \tag{3}$$

To solve for how the electric field would look like at a different propagation distance we first decompose the initial electric field into a *superposition* of these eigenmodes at $z = 0$,

$$E(x, y, z = 0, t) = \int \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} \tilde{E}(k_x, k_y) \exp(jk_x x + jk_y y) \exp(-j\omega_0 t) \tag{4}$$

We see that the initial mode profile in x and y is related to the eigenmode amplitudes $\tilde{E}(k_x, k_y)$ through a Fourier transform,

$$\tilde{E}(k_x, k_y) = \int dx \int dy E(x, y, z = 0, t) \exp(-jk_x x - jk_y y) \tag{5}$$

The field profile at a distance $z = L$ is then given by

$$E(x, y, z, t) = \int \frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\tilde{E}(k_x, k_y)\exp(jk_x x + jk_y y - j\frac{k_x^2 + k_y^2}{2k_0}z)\exp(jk_0 z - j\omega_0 t) \qquad (6)$$

This is, of course, Fresnel diffraction. Coming back to our Gaussian beam, we first solve for the $\tilde{E}(k_x, k_y)$ from the initial beam profile,

$$\tilde{E}(k_x, k_y) = \pi W_0^2 E_0 \exp\Big[-\frac{W_0^2(k_x^2 + k_y^2)}{2}\Big] \qquad (7)$$

Calculating the Fresnel diffraction, we have

$$E(x, y, z, t) = E_0 \frac{1}{1 + jz/(k_0 W_0^2)} \exp\Big[-\frac{x^2 + y^2}{2W_0^2(1 + jz/(k_0 W_0^2))}\Big]\exp(jk_0 z - j\omega_0 t) \qquad (8)$$

Looking at the formula for the Gaussian beam diffraction, If we have $z << k_0 W_0^2$, then the beam would not look too different from the initial profile. Otherwise, if z is on the order of $k_0 W_0^2$, diffraction starts to have a pronounced effect on the beam. $k_0 W_0^2$ is called the Rayleigh length, which is the propagation distance at which diffraction kicks in.

GROUP VELOCITY

To go around obstacles, it is better to confine light in waveguides. Consider a waveguide with only one single mode. Let's say this single mode profile is given by

$$\mathbf{F}(x, y)\exp\big[j\beta(\omega)z - j\omega t\big] \qquad (9)$$

One important point is that the mode beam profile $\mathbf{F}(x, y)$ is usually not the same as your input beam, so, for example, if you try to shine a Gaussian beam on one end of fiber, not all the energy of the Gaussian beam is going into the mode. The rest of the energy is coupled to other modes, but since this is a single-mode waveguide, the rest of the modes are *lossy* (β has an imaginary component).

The second important point is that the propagation constant, β, depends on frequency. This dependence can come from

1. The material of the waveguide itself (somehow the material has different refractive index for different frequencies)

2. The waveguide structure (different constructive inteference conditions for different frequencies)

For example, for a metal waveguide, β is given by

$$\beta = \frac{n\omega}{c}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \tag{10}$$

where $n = n(\omega)$ is the refractive index, which can be a function of the frequency due to the material inside the metal waveguide.

Now let's say we perform *amplitude modulation* on a laser field at $z = 0$ and couple that into a waveguide,

$$E(t) = A(t)\exp(-j\omega_0 t) \tag{11}$$

where $A(t)$ is my signal. In the frequency domain, I have

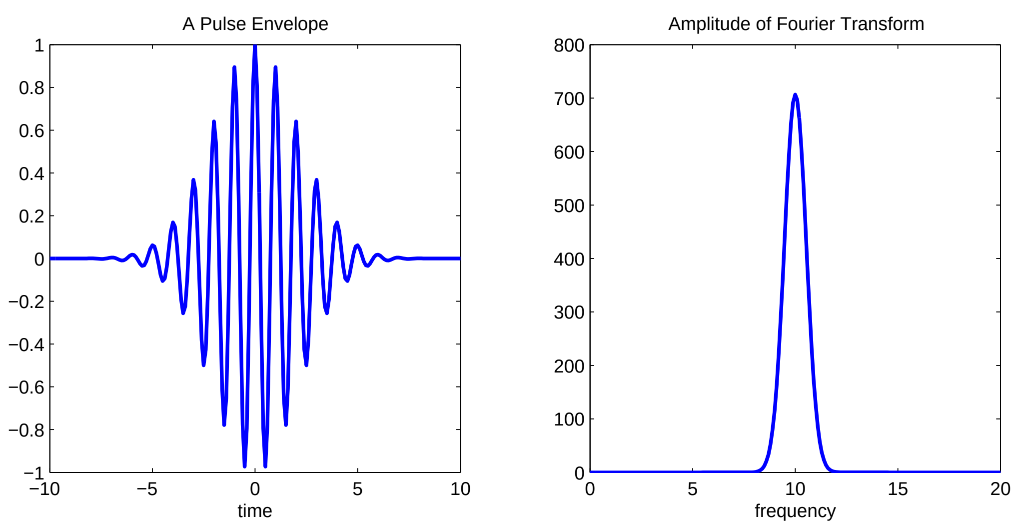
$$\tilde{E}(\omega) = \int dt E(t)\exp(j\omega t) \tag{12}$$

$$= \int dt A(t)\exp(-j\omega_0 t)\exp(j\omega t) \tag{13}$$

$$= \int dt A(t)\exp\left[j(\omega - \omega_0)t\right] \tag{14}$$

$$= \tilde{A}(\omega - \omega_0) \tag{15}$$

If $A(t)$ is a Gaussian, I would have something that looks like this



To find out what the pulse would look like at the other end of the fiber, I have

$$E(z,t) = \int \frac{d\omega}{2\pi}\tilde{A}(\omega - \omega_0)\exp\left[j\beta(\omega)z - j\omega t\right] \tag{16}$$

If the bandwidth of $\tilde{A}$ is much smaller than ω_0, it is only going to "see" a small portion of $\beta(\omega)$ near ω_0. I can then approximate $\beta(\omega)$ by a Taylor series near ω_0:

$$\beta(\omega) = \beta_0 + \beta_1 \times (\omega - \omega_0) + \ldots \tag{17}$$

where β_0 is $\beta(\omega_0)$, and $\beta_1 = \beta'(\omega_0)$. If we simply stop at the first-order approximation, we have

$$E(z,t) = \int \frac{d\omega}{2\pi} \tilde{A}(\omega - \omega_0) \exp\Big[j\beta_0 z + j\beta_1(\omega - \omega_0)z - j\omega t \Big] \tag{18}$$

$$= \int \frac{d\omega}{2\pi} \tilde{A}(\omega - \omega_0) \exp\Big[-j(\omega - \omega_0)(t - \beta_1 z) \Big] \exp(j\beta_0 z - j\omega_0 t) \tag{19}$$

$$= A(t - \beta_1 z) \exp(j\beta_0 z - j\omega_0 t) \tag{20}$$

We see that the envelope propagates at a velocity $1/\beta_1 = d\omega/d\beta$, while the phase propagates at a different velocity ω_0/β_0. The envelope velocity is called the *group velocity*,

$$v_g \equiv \frac{1}{\beta_1} = \frac{d\omega}{d\beta} \tag{21}$$

while the velocity at which the ripples travel is called the *phase velocity*,

$$v_p = \frac{\omega_0}{\beta_0} \tag{22}$$

A video would show this more clearly:

GROUP-VELOCITY DISPERSION

Now, if the bandwidth of $\tilde{A}$ is a little higher so that it is not enough to approximate β with a linear function, we will have to expand it to a higher order,

$$\beta(\omega) \approx \beta_0 + \beta_1 \times (\omega - \omega_0) + \frac{\beta_2}{2} \times (\omega - \omega_0)^2 \tag{23}$$

The wave function at z is

$$E(z,t) = \int \frac{d\omega}{2\pi} \tilde{A}(\omega - \omega_0) \exp\Big[j\beta_0 z + j\beta_1(\omega - \omega_0)z + j\frac{\beta_2}{2}(\omega - \omega_0)^2 z - j\omega t \Big] \tag{24}$$

A change of variable $\omega' = \omega - \omega_0$ will make this look less cumbersome

$$E(z,t) = \int \frac{d\omega'}{2\pi} \tilde{A}(\omega') \exp\Big[j\beta_0 z + j\frac{\beta_2}{2}\omega'^2 z - j\omega'(t - \beta_1 z) \Big] \exp(-j\omega_0 t) \tag{25}$$

But we know how to solve this. This has the exact some form as 1D Fresnel diffraction. Again take our favorite Gaussian as an example, we have as our final result

$$E(z,t) = \frac{E_0}{\sqrt{1 - jz/(T_0^2/\beta_2)}} \exp\Big\{ -\frac{(t - \beta_1 z)^2}{2T_0^2[1 - jz/(T_0^2/\beta_2)]} \Big\} \exp(j\beta_0 z - j\omega_0 t) \tag{26}$$

This is very similar to diffraction. For $z << |T_0^2/\beta_2|$, group-velocity dispersion doesn't have much effect. The effect only kicks in when z is on the order of $|T_0^2/\beta_2|$, which is called the *dispersion length*, analogous to the Rayleigh length in diffraction.

There is one crucial difference between ordinary diffraction and group-velocity dispersion. For group-velocity dispersion, β_2 can have a positive or negative sign. This means that if I have one chunk of waveguide with a positive β_2, and a chunk of waveguide with a negative β_2, the dispersion of the two waveguides can cancel each other. See Problem 12.30, Hayt and Buck (6th edition).